数理解析研究所講究録
第2014巻 2017年 67-78

A mean value on the sum of two primes in arithmetic progressions

Yuta Suzuki

1 Introduction

In this note, we consider the sum of two primes in arithmetic progressions, which is a generalization of the Goldbach problem. Although it is somewhat an indirect way, we shall consider this problem in some average sense as Rüppel did in [12, 13]. For this additive problem, we use the representation function given by

$$R(n,\mathbf{q},\mathbf{a})=R(n,q_1,a_1,q_2,a_2):=\sum_{\substack{m_1+m_2=n\\ m_i\equiv a_i \pmod{q_i}}}\Lambda(m_1)\Lambda(m_2),$$

where $\Lambda(n)$ is the von Mangoldt function, a_1,a_2,q_1,q_2,n are positive integers satisfying $(a_1,q_1)=(a_2,q_2)=1$ and $\mathbf{q}=(q_1,q_2),\mathbf{a}=(a_1,a_2)$. Let us also introduce

$$R(n,q,a,b):=R(n,q,a,q,b)$$

for positive integers a,b,q satisfying $(ab,q)=1$. In 2009, Rüppel [12] studied the mean value of this representation function, i.e.

$$\sum_{n\leq X}R(n,q,a,b). \tag{1}$$

In particular, she obtained, under a weakened variant of the Generalized Riemann Hypothesis (GRH), an asymptotic formula for the mean value (1). More precisely, she assumed GRH except the existence of real zeros. In this note, we assume a hypothesis similar to, but even weaker than, Rüppel's one.

Hypothesis (GRH with real zeros). *Every complex non-trivial zero of Dirichlet L functions in the strip $0<\sigma<1$ lies on the critical line $\sigma=1/2$.*

We call this hypothesis GRHR as an abbreviation.

Assuming this hypothesis GRHR, Rüppel [12] proved

$$\sum_{n\leq X}R(n,q,a,b)=\frac{X^2}{2\varphi(q)^2}+O\left(X^{1+\delta}(\log q)^2\right), \tag{2}$$

where $\delta=1/2$ unless real zeros exist for the modulus q, in which case we let δ be the largest one among these real zeros. She considered the mean value (1), but we can also obtain the corresponding result for the mean value

$$\sum_{n\le X} R(n, q_1, a_1, q_2, a_2) \tag{3}$$

by the same method. Moreover, her method can be used to prove

$$\begin{aligned}\sum_{n\le X} R(n,q,a,b) &= \frac{X^2}{2\varphi(q)^2} - \sum_{\chi \pmod q} \frac{\overline{\chi}(a)+\overline{\chi}(b)}{\varphi(q)^2} \sum_{\beta_\chi} \frac{X^{\beta_\chi+1}}{\beta_\chi(\beta_\chi+1)} \\ &+ \sum\sum_{\chi,\psi \pmod q} \frac{\overline{\chi}(a)\overline{\psi}(b)}{\varphi(q)^2} \sum\sum_{\beta_\chi,\beta_\psi} \frac{\Gamma(\beta_\chi)\Gamma(\beta_\psi)}{\Gamma(\beta_\chi+\beta_\psi)} \frac{X^{\beta_\chi+\beta_\psi}}{\beta_\chi+\beta_\psi} + O\left(X^{3/2}(\log q)^2\right),\end{aligned} \tag{4}$$

where β_χ runs through all real zeros of $L(s,\chi)$ with $\beta_\chi > 1/2$.

These results correspond to the result of Fujii [4] for the ordinary Goldbach Problem. Fujii [4] proved, for the representation function

$$R(n) = \sum_{m_1+m_2=n} \Lambda(m_1)\Lambda(m_2)$$

of the ordinary Goldbach problem, an asymptotic formula

$$\sum_{n\le X} R(n) = \frac{X^2}{2} + O\left(X^{3/2}\right)$$

under the Riemann Hypothesis (RH). This was improved by Fujii [5] himself to

$$\sum_{n\le X} R(n) = \frac{X^2}{2} - 2\sum_{\rho} \frac{X^{\rho+1}}{\rho(\rho+1)} + O\left((X\log X)^{4/3}\right) \tag{5}$$

under RH, where ρ runs through all non-trivial zeros of the Riemann zeta function. After this pioneering work of Fujii, the error term on the right-hand side of (5) was improved to

$$\begin{aligned}&\ll X(\log X)^5 \quad \text{(by Bhowmik and Schlage-Puchta [1])},\\ &\ll X(\log X)^3 \quad \text{(by Languasco and Zaccagnini [9])}.\end{aligned}$$

Of course we assume RH in all of these results. We note that Bhowmik and Schlage-Puchta proved also the omega result

$$= \Omega(X \log\log X) \tag{6}$$

for this error term. This omega result is independent from RH or GRH.

In [17], we improved Rüppel's result (2) up to the accuracy of the result of Languasco and Zaccagnini [9]. For a given Dirichlet character $\chi \pmod q$, we introduce the set of "non-trivial zeros" of $L(s,\chi)$

$$\mathcal{Z}(\chi) := \begin{cases} \{\, \rho = \beta + i\gamma \mid 0<\beta<1,\ \zeta(\rho)=0 \,\} \cup \{1\} & \text{(when } \chi \text{ is principal)}, \\ \{\, \rho = \beta + i\gamma \mid 0<\beta<1,\ L(\rho,\chi)=0 \,\} & \text{(otherwise)} \end{cases}$$

and $\mathcal{Z}_0(\chi) := \mathcal{Z}(\chi) \cap (1/2, 1]$. For $\rho \in \mathcal{Z}(\chi)$, we let

$$A(\rho) := \begin{cases} 1 & (\text{when } \rho = 1), \\ -m(\rho) & (\text{when } \rho \neq 1), \end{cases}$$

where $m(\rho)$ is the multiplicity of ρ as a zero of $L(s,\chi)$. For $\Re\kappa$, $\Re\mu > 0$ and $X \geq 2$, we let

$$W_0(\kappa,\mu) = W_0(X,\kappa,\mu) := \frac{\Gamma(\kappa)\Gamma(\mu)}{\Gamma(\kappa+\mu)} \frac{X^{\kappa+\mu}}{\kappa+\mu},$$

and for $\rho_1 \in \mathcal{Z}(\chi_1)$, $\rho_2 \in \mathcal{Z}(\chi_2)$ and $X \geq 2$, we let

$$W(\rho_1,\rho_2) = W(X,\rho_1,\rho_2) := A(\rho_1)A(\rho_2)W_0(X,\rho_1,\rho_2).$$

Throughout this note, $E(X)$ denotes error terms which can be estimated as

$$E(X) \ll X(\log X)(\log q_1 X)(\log q_2 X).$$

Then the main result in [17] is:

Theorem 1. *Assume GRHR. For $X \geq 2$, we have*

$$\sum_{n \leq X} R(n,\mathbf{q},\mathbf{a}) = \sum_{\substack{\chi_1 \pmod{q_1} \\ \chi_2 \pmod{q_2}}}\sum \frac{\overline{\chi_1}(a_1)\overline{\chi_2}(a_2)}{\varphi(q_1)\varphi(q_2)} \sum_{\substack{\rho_1 \in \mathcal{Z}(\chi_1) \\ \rho_2 \in \mathcal{Z}(\chi_2) \\ \beta_1+\beta_2>1}}\sum W(\rho_1,\rho_2) + E(X),$$

where the implicit constant in the error term is absolute.

The main aim of this note is to give an alternative proof of Theorem 1.

We remark that the main term of Rüppel's result (2) appears in Theorem 1 as the term corresponding to the pair $(1,1) \in \mathcal{Z}(\chi_0) \times \mathcal{Z}(\chi_0)$. Note that $W(1,1) = X^2/2$. Moreover, the main and oscillating terms on the right-hand side of (4) correspond to the pairs of "real non-trivial zeros" in $\mathcal{Z}_0(\chi_1) \times \mathcal{Z}_0(\chi_2)$. The other terms in Theorem 1 was included in the error terms of (2) and (4). Hence Theorem 1 is an improvement of these results (2) and (4) of Rüppel.

Our previous proof [17] of Theorem 1 follows the argument of Languasco and Zaccagnini [9], which uses the circle method with power series. Recently, Goldston and Yang succeeded in obtaining the result of Languasco and Zaccagnini even via finite trigonometric polynomials. The key point of their argument is a quite neat preliminary averaging which will be carried out in (8). This preliminary averaging enables us to save some logarithms from the estimate of $K(U,h)$ defined by (19). See the proof of Lemma 3. In this note, we sketch an alternative proof of Theorem 1 by using this technique of Goldston and Yang [7].

2 The circle method and the Goldston-Yang trick

Instead of the original representation function $R(n,\mathbf{q},\mathbf{a})$, we use the twisted representation function

$$R(n,\chi_1,\chi_2) := \sum_{m_1+m_2=n} \chi_1(m_1)\Lambda(m_1)\chi_2(m_2)\Lambda(m_2)$$

for Dirichlet characters $\chi_1 \pmod{q_1}$, $\chi_2 \pmod{q_2}$. We use two trigonometric polynomials

$$S(\alpha, U, \chi) := \sum_{n \le U} \chi(n)\Lambda(n)e(n\alpha),$$

$$T_\beta(\alpha, U) := \sum_{n \le U} n^{\beta-1}e(n\alpha), \quad T(\alpha) := T_1(\alpha, X),$$

where $e(\alpha) := \exp(2\pi i\alpha)$, $U \ge X$ and $\chi \pmod{q}$ is a Dirichlet character. Clearly,

$$\sum_{n \le X} R(n, \chi_1, \chi_2) = \int_{-1/2}^{1/2} S(\alpha, U, \chi_1)S(\alpha, U, \chi_2)T(-\alpha)d\alpha. \tag{7}$$

Here we note that this equation (7) holds for any $U \ge X$. Hence we can take an average over $X \le U \le 2X$. Then we have

$$\sum_{n \le X} R(n, \chi_1, \chi_2) = \frac{1}{X}\int_X^{2X}\int_{-1/2}^{1/2} S(\alpha, U, \chi_1)S(\alpha, U, \chi_2)T(-\alpha)d\alpha\, dU. \tag{8}$$

We introduce the approximation

$$S(\alpha, U, \chi) = \sum_{\beta \in \mathcal{Z}_0(\chi)} A(\beta)T_\beta(\alpha, U) + R(\alpha, U, \chi).$$

Then the integral in (8) can be expanded as

$$\begin{aligned} &= \sum_{\beta_1 \in \mathcal{Z}_0(\chi_1)} A(\beta_1)I_{\beta_1 S_2} + \sum_{\beta_2 \in \mathcal{Z}_0(\chi_2)} A(\beta_2)I_{\beta_2 S_1} \\ &\quad - \sum_{\substack{\beta_1 \in \mathcal{Z}_0(\chi_1) \\ \beta_2 \in \mathcal{Z}_0(\chi_2)}}\sum A(\beta_1)A(\beta_2)I_{\beta_1\beta_2} + I_R, \end{aligned}$$

where

$$I_{\beta_i S_j} := \frac{1}{X}\int_X^{2X}\int_{-1/2}^{1/2} T_{\beta_i}(\alpha, U)S(\alpha, U, \chi_j)T(-\alpha)d\alpha\, dU, \tag{9}$$

$$I_{\beta_1\beta_2} := \frac{1}{X}\int_X^{2X}\int_{-1/2}^{1/2} T_{\beta_1}(\alpha, U)T_{\beta_2}(\alpha, U)T(-\alpha)d\alpha\, dU, \tag{10}$$

$$I_R := \frac{1}{X}\int_X^{2X}\int_{-1/2}^{1/2} R(\alpha, U, \chi_1)R(\alpha, U, \chi_2)T(-\alpha)d\alpha\, dU. \tag{11}$$

3 The estimate of Bhowmik and Schlage-Puchta

We first estimate the integral (11). This kind of estimate without characters was obtained by Bhowmik and Schlage-Puchta [1] and improved by Languasco and

Zaccagnini [9]. We have to generalize their result to the case of the integral with Dirichlet characters. Originally, the improvement of Languasco and Zaccagnini [9] was based on the circle method with power series, which is also applicable to our integral with characters. However, as we mentioned in Section 1, our exposition follows an alternative method of Goldston and Yang [7], which uses only finite trigonometric polynomials instead of power series.

By the Cauchy-Schwarz inequality, we have

$$I_R \ll J(\chi_1)^{1/2} J(\chi_2)^{1/2}, \tag{12}$$

where

$$J(\chi) = \int_{-1/2}^{1/2} E(\alpha, \chi)\, |T(\alpha)|\, d\alpha \tag{13}$$

and

$$E(\alpha, \chi) := \frac{1}{X} \int_X^{2X} |R(\alpha, U, \chi)|^2\, dU. \tag{14}$$

Note that $T(\alpha) \ll \min(X, 1/|\alpha|)$ for $|\alpha| \le 1/2$. Hence we obtain

$$J(\chi) \ll X \int_{|\alpha| \le 1/X} E(\alpha, \chi) d\alpha + \int_{1/X < |\alpha| \le 1/2} E(\alpha, \chi) \frac{d\alpha}{|\alpha|}.$$

The latter integral is

$$= \int_{1/X < |\alpha| \le 1/2} E(\alpha, \chi) \left(2 + \int_{|\alpha|}^{1/2} \frac{d\xi}{\xi^2} \right) d\alpha$$

$$\ll \int_{|\alpha| \le 1/2} E(\alpha, \chi) d\alpha + \int_{1/X < \xi \le 1/2} \left(\int_{|\alpha| \le \xi} E(\alpha, \chi) d\alpha \right) \frac{d\xi}{\xi^2}.$$

Thus we have

$$J(\chi) \ll (\log X) \sup_{1/X \le \xi \le 1/2} \frac{1}{\xi} \int_{|\alpha| \le \xi} E(\alpha, \chi) d\alpha. \tag{15}$$

As a result our estimate is reduced to the estimate of the integral

$$\int_{|\alpha| \le \xi} E(\alpha, \chi) d\alpha = \frac{1}{X} \int_X^{2X} \int_{|\alpha| \le \xi} |R(\alpha, U, \chi)|^2 d\alpha\, dU. \tag{16}$$

By Gallagher's lemma [6, Lemma 1], we have

$$\int_{|\alpha| \le \xi} |R(\alpha, U, \chi)|^2 d\alpha \ll \xi^2 \int_{-\infty}^{\infty} \left| \sideset{}{^\sharp}\sum_{\substack{x < n \le x + (2\xi)^{-1} \\ n \le U}} \chi(n)\Lambda(n) \right|^2 dx, \tag{17}$$

where

$$\sideset{}{^\sharp}\sum_n \chi(n)\Lambda(n) := \sum_n \chi(n)\Lambda(n) - \sum_{\beta \in \mathcal{Z}_0(\chi)} A(\beta) \sum_n n^{\beta - 1}.$$

Now we separate some irregular parts of the integral on the right-hand of (17):

$$\int_{-\infty}^{\infty} \Bigg| \sideset{}{^\sharp}\sum_{\substack{x<n\le x+(2\xi)^{-1} \\ n\le U}} \chi(n)\Lambda(n) \Bigg|^2 dx \tag{18}$$
$$\ll I((2\xi)^{-1}) + J(U,(2\xi)^{-1}) + K(U,(2\xi)^{-1}),$$

where

$$I(h) := \int_0^h \Bigg| \sideset{}{^\sharp}\sum_{n\le x} \chi(n)\Lambda(n) \Bigg|^2 dx, \quad J(U,h) := \int_0^U \Bigg| \sideset{}{^\sharp}\sum_{x<n\le x+h} \chi(n)\Lambda(n) \Bigg|^2 dx,$$

$$K(U,h) := \int_{U-h}^{U} \Bigg| \sideset{}{^\sharp}\sum_{U<n\le x+h} \chi(n)\Lambda(n) \Bigg|^2 dx, \tag{19}$$

and we assume $1 \le h \le X/2$ in what follows. We estimate these integrals. First we recall the follwoing explicit formula which can be derived from Theorem 12.5 and Theorem 12.10 of [10]:

$$\psi(x,\chi) = \sum_{\substack{\rho\in\mathcal{Z}(\chi) \\ |\gamma|\le T}} A(\rho)\frac{x^\rho}{\rho} + \frac{L'}{L}(1,\overline{\chi}) + O\left(\log qX\right), \tag{20}$$

where $2 \le x \le X$, $T = X(\log qX)$ and χ is primitive. Moreover, we need the following lemma:

Lemma 1. *For any non-principal character $\chi \pmod q$, we have*

$$\frac{L'}{L}(1,\overline{\chi}) = \frac{1}{1-\tilde{\beta}} + O(\log q), \tag{21}$$

$$\sum_{\beta\ne 1-\tilde{\beta}} \frac{1}{\beta} = \sum_{\beta\ne\tilde{\beta}} \frac{1}{1-\beta} \ll \log q, \tag{22}$$

$$\frac{1}{1-\tilde{\beta}} \ll q^{1/2}(\log q)^2, \tag{23}$$

where β runs thorough all real zeros of $L(s,\chi)$ counted with multiplicity and $\tilde{\beta}$ is the possible exceptional zero for $\chi \pmod q$.

Proof. For the assertion (21), see Theorem 11.4 [10]. Note that

$$\sum_{\beta\ne\tilde{\beta}} \frac{1}{1-\beta} \le \sum_{\rho\ne\tilde{\beta}} \Re\left(\frac{1}{1-\rho}\right).$$

Then the estimate (22) can be obtained via the proof of (11.13) of [10]. The last estimate (23) is the famous estimate of Page, see Corollary 11.12 of [10]. □

The estimate for the integral $I(h)$ is essentially the famous result of Cramér. For the original result of Cramér, see [2] or [10, Theorem 13.5].

Lemma 2. *Assume GRHR. Then we have* $I(h) \ll h^2(\log 2q)^2$.

Proof. (*Sketch*) Combining (20) with Lemma 1, we have

$$\sum_{n\le x}^{\sharp} \chi(n)\Lambda(n) = \sum_{0<|\gamma|\le T} \frac{x^\rho}{\rho} + O\left(h^{1/2}\log 2q\right),$$

where $2 \le x \le h$ and $T = h\log qh$. Hence it is sufficient to estimate the integral

$$\int_2^h \left| \sum_{0<|\gamma|\le T} \frac{x^\rho}{\rho} \right|^2 dx.$$

By expanding square and using GRHR, we have this integral is

$$\begin{aligned} &\ll \sum_{0<|\gamma_1|\le T}\sum_{0<|\gamma_2|\le T} \frac{h^2}{(1+|\gamma_1|)(1+|\gamma_2|)}\cdot\frac{1}{1+|\gamma_1-\gamma_2|} \\ &\ll h^2(\log 2q)\sum_{0<|\gamma_1|\le T}\frac{(\log(1+|\gamma_1|))^2}{1+|\gamma_1|^2} \ll h^2(\log 2q)^2. \end{aligned}$$

This proves the lemma. □

The integral $K(U,h)$, although it seems to be not so important at first sight, is quite annoying part to estimate, which was a main obstacle to reduce the logarithms from the final error term as Languasco and Zaccagnini [9] did. This is exactly the place where we launch the weapon of Goldston and Yang [7].

Lemma 3. *Assume GRHR. Then we have*

$$\frac{1}{X}\int_X^{2X} K(U,h)dU \ll hX(\log q)^2.$$

Proof. The left-hand side of the assertion is

$$\begin{aligned} &= \frac{1}{X}\int_X^{2X}\int_U^{U+h} \left|\sum_{U<n\le x}^{\sharp} \chi(n)\Lambda(n)\right|^2 dx\, dU \\ &\ll \frac{h}{X}\int_X^{2X} \left|\sum_{n\le U}^{\sharp}\chi(n)\Lambda(n)\right|^2 dU + \frac{h}{X}\int_X^{3X}\left|\sum_{n\le x}^{\sharp}\chi(n)\Lambda(n)\right|^2 dx. \end{aligned}$$

Therefore the lemma immediately follows from Lemma 2 □

The estimate of the "main-part" integral $J(U,h)$ is classical. Actually, the special case without Dirichlet characters was already obtained by Selberg [15]. The estimate necessary for Theorem 1 is obtained via the techniques of Saffari and Vaughan [14]. See also Prachar [11] or Yuan and Zun [16].

Lemma 4. *Assume GRHR. Then we have* $J(U,h) \ll hU(\log qU)^2$.

Proof. (*Sketch*) We first decompose the integral as

$$J(U,h) = \int_0^h + \int_h^U = J_1 + J_2, \text{ say.}$$

The first integral J_1 is estimated by Lemma 2 as

$$J_1 \ll h^2(\log 2q)^2 \ll hU(\log qU)^2,$$

which is admissible. The integral J_2 can be estimated by the techniques of Saffari and Vaughan [14] as

$$J_2 \ll hU(\log qU)^2.$$

For the details, see Lemma 5 and Lemma 6 of [14]. This completes the proof. □

Applying (18), Lemma 2 and Lemma 4 to (17), we have

$$\int_{|\alpha|\le\xi} |R(\alpha,U,\chi)|^2 d\alpha \ll \xi X(\log qX)^2 + \xi^2 K(U,(2\xi)^{-1}).$$

Hence by (15), (16) and Lemma 3, we have

$$J(\chi) \ll X(\log X)(\log qX)^2.$$

Substituting this estimate into (12), we arrive at

$$I_R \ll E(X). \tag{24}$$

This completes the estimate of I_R.

4 Detection of main and oscillating terms

We next calculate the integral $I_{\beta_i S_j}$ defined by (9). Clearly,

$$I_{\beta_i S_j} = \int_{-1/2}^{1/2} T_{\beta_i}(\alpha,X)S(\alpha,X,\chi_j)T(-\alpha)d\alpha = \sum_{m+n\le X} \chi_j(m)\Lambda(m)n^{\beta_i-1}$$

We now calculate this last sum explicitly. The result is

Lemma 5. *Assume GRHR. For any real numbers* $X \ge 2$ *and* $1/2 < \mu \le 1$, *we have*

$$\sum_{m+n\le X} \chi(m)\Lambda(m)n^{\mu-1} = \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \beta+\mu>1}} A(\rho)W_0(\rho,\mu) + O(XQ(\chi)(\log 2q)(\log X)),$$

where

$$Q(\chi) := \begin{cases} q^{1/2}(\log 2q) & (\textit{when } \chi \text{ (mod } q) \textit{ is an exceptional character}), \\ 1 & (\textit{otherwise}). \end{cases}$$

Proof. Recalling $1/2 < \mu \leq 1$, we have

$$\begin{aligned}\sum_{m+n\leq X} \chi(m)\Lambda(m)n^{\mu-1} &= \sum_{m\leq X} \chi(m)\Lambda(m) \sum_{n\leq X-m} n^{\mu-1} \\ &= \sum_{m\leq X} \chi(m)\Lambda(m) \int_0^{X-m} u^{\mu-1}du + O(X) \\ &= \int_0^X \psi(u,\chi)(X-u)^{\mu-1}du + O(X).\end{aligned}$$

Now we use the explicit formula (20). We can remove the restriction that $\chi \pmod q$ is primitive with the error of the size $(\log 2q)(\log X)$. By (20), we have

$$\begin{aligned}\sum_{m\leq X} \psi_\mu(m,\chi) &= \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ |\gamma|\leq T}} \frac{A(\rho)}{\rho} \int_0^X u^\rho (X-u)^{\mu-1}du + \frac{X^\mu}{\mu}\cdot\frac{L'}{L}(1,\overline{\chi}) + E_1(X) \\ &= \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ |\gamma|\leq T}} A(\rho)W_0(\rho,\mu) + \frac{X^\mu}{\mu}\cdot\frac{L'}{L}(1,\overline{\chi}) + E_1(X),\end{aligned}$$

where $E_1(X) \ll X(\log 2q)(\log X)$. Now we remove the restriction $|\gamma| \leq T$ from the above sum over $\mathcal{Z}(\chi)$. Note that Stirling's formula implies

$$W_0(\rho,\mu) = \frac{\Gamma(\rho)\Gamma(\mu)}{\Gamma(\rho+\mu)}\frac{X^{\rho+\mu}}{\rho+\mu} \ll \frac{X^{\beta+\mu}}{(1+|\gamma|)^{1+\mu}}\left(1+\frac{1}{|\rho|}\right), \tag{25}$$

where we have to care about the pole $\rho = 0$ of the factor $\Gamma(\rho)$. Then we can estimate the extended part of the sum by

$$\ll X^{\mu+1} \sum_{\substack{\rho\\ |\gamma|>T}} \frac{1}{|\gamma|^{1+\mu}} \ll \frac{X^{\mu+1}}{T^\mu}(\log qT) \ll X(\log 2q)(\log X).$$

Hence we obtain

$$\sum_{m\leq X} \psi_\mu(m,\chi) = \sum_{\rho\in\mathcal{Z}(\chi)} A(\rho)W_0(\rho,\mu) + \frac{X^\mu}{\mu}\cdot\frac{L'}{L}(1,\overline{\chi}) + E_1(X). \tag{26}$$

By (21) and (23), we have

$$\frac{X^\mu}{\mu}\cdot\frac{L'}{L}(1,\overline{\chi}) \ll X^\mu Q(\chi)(\log 2q) \ll XQ(\chi)(\log 2q). \tag{27}$$

By using (22), (23), (25) and GRHR, we have

$$\begin{aligned}\sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \beta+\mu\leq 1}} A(\rho)W_0(\rho,\mu) &\ll X\left(\sum_\beta \frac{1}{\beta} + \sum_\rho \frac{1}{(1+|\gamma|)^{3/2}}\right) \\ &\ll XQ(\chi)(\log 2q).\end{aligned} \tag{28}$$

Substituting (27) and (28) into (26), we obtain the lemma. □

Finally, the integral $I_{\beta_1\beta_2}$ (10) is obviously

$$\begin{aligned} I_{\beta_1\beta_2} &= \int_{-1/2}^{1/2} T_{\beta_1}(\alpha, X)T_{\beta_2}(\alpha, X)T(-\alpha)d\alpha \\ &= \sum_{m+n\le X} m^{\beta_1-1}n^{\beta_2-1} = W_0(\beta_1,\beta_2) + O(X). \end{aligned} \tag{29}$$

5 Completion of the proof

By Lemma 5 and (29), we have

$$\begin{aligned} &\sum_{\beta_1\in\mathcal{Z}_0(\chi_1)} A(\beta_1)I_{\beta_1 S_2} + \sum_{\beta_2\in\mathcal{Z}_0(\chi_2)} A(\beta_2)I_{\beta_2 S_1} - \sum_{\substack{\beta_1\in\mathcal{Z}_0(\chi_1)\\ \beta_2\in\mathcal{Z}_0(\chi_2)}}\sum A(\beta_1)A(\beta_2)I_{\beta_1\beta_2} \\ &\qquad = \sum_{\substack{\beta_1\in\mathcal{Z}_0(\chi_1)\\ \rho_2\in\mathcal{Z}(\chi_2)\\ \beta_1+\beta_2>1}}\sum W(\beta_1,\rho_2) + \sum_{\substack{\rho_1\in\mathcal{Z}(\chi_1)\\ \beta_2\in\mathcal{Z}_0(\chi_2)\\ \beta_1+\beta_2>1}}\sum W(\rho_1,\beta_2) \\ &\qquad - \sum_{\substack{\beta_1\in\mathcal{Z}_0(\chi_1)\\ \beta_2\in\mathcal{Z}_0(\chi_2)}}\sum W(\beta_1,\beta_2) + (Q(\chi_1)+Q(\chi_2))E(X). \end{aligned}$$

Since the condition $\beta_1+\beta_2>1$ implies that $\beta_1>1/2$ or $\beta_2>1/2$, we find that

$$\begin{aligned} &\{\,(\rho_1,\rho_2)\in\mathcal{Z}(\chi_1)\times\mathcal{Z}(\chi_2)\mid \beta_1+\beta_2>1\,\} \\ &\subset \mathcal{Z}_0(\chi_1)\times\mathcal{Z}(\chi_2)\cup\mathcal{Z}(\chi_1)\times\mathcal{Z}_0(\chi_2) \end{aligned}$$

and that

$$\mathcal{Z}_0(\chi_1)\times\mathcal{Z}(\chi_2)\cap\mathcal{Z}(\chi_1)\times\mathcal{Z}_0(\chi_2) = \mathcal{Z}_0(\chi_1)\times\mathcal{Z}_0(\chi_2).$$

Therefore

$$\sum_{\substack{\beta_1\in\mathcal{Z}_0(\chi_1)\\ \rho_2\in\mathcal{Z}(\chi_2)\\ \beta_1+\beta_2>1}}\sum + \sum_{\substack{\rho_1\in\mathcal{Z}(\chi_1)\\ \beta_2\in\mathcal{Z}_0(\chi_2)\\ \beta_1+\beta_2>1}}\sum = \sum_{\substack{\rho_1\in\mathcal{Z}(\chi_1)\\ \rho_2\in\mathcal{Z}(\chi_2)\\ \beta_1+\beta_2>1}}\sum + \sum_{\substack{\beta_1\in\mathcal{Z}_0(\chi_1)\\ \beta_2\in\mathcal{Z}_0(\chi_2)}}\sum .$$

Thus we have

$$\begin{aligned} &\sum_{\beta_1\in\mathcal{Z}_0(\chi_1)} A(\beta_1)I_{\beta_1 S_2} + \sum_{\beta_2\in\mathcal{Z}_0(\chi_2)} A(\beta_2)I_{\beta_2 S_1} - \sum_{\substack{\beta_1\in\mathcal{Z}_0(\chi_1)\\ \beta_2\in\mathcal{Z}_0(\chi_2)}}\sum A(\beta_1)A(\beta_2)I_{\beta_1\beta_2} \\ &\qquad = \sum_{\substack{\rho_1\in\mathcal{Z}(\chi_1)\\ \rho_2\in\mathcal{Z}(\chi_2)\\ \beta_1+\beta_2>1}}\sum W(\rho_1,\rho_2) + (Q(\chi_1)+Q(\chi_2))E(X). \end{aligned}$$

Substituting this equation and (24) into the result of Section 2, we obtain

$$\sum_{n\leq X} R(n,\chi_1,\chi_2) = \sum_{\substack{\rho_1\in\mathcal{Z}(\chi_1)\\ \rho_2\in\mathcal{Z}(\chi_2)\\ \beta_1+\beta_2>1}}\sum W(\rho_1,\rho_2) + (Q(\chi_1)+Q(\chi_2))E(X).$$

By the orthogonality of Dirichlet characters, we have

$$\sum_{n\leq X} R(n,\mathbf{q},\mathbf{a}) = \sum_{\substack{\chi_1 \pmod{q_1}\\ \chi_2 \pmod{q_2}}}\sum \frac{\overline{\chi_1}(a_1)\overline{\chi_2}(a_2)}{\varphi(q_1)\varphi(q_2)} \sum_{\substack{\rho_1\in\mathcal{Z}(\chi_1)\\ \rho_2\in\mathcal{Z}(\chi_2)\\ \beta_1+\beta_2>1}}\sum W(\rho_1,\rho_2) + Q\cdot E(X), \tag{30}$$

where

$$Q := \sum_{\substack{\chi_1 \pmod{q_1}\\ \chi_2 \pmod{q_2}}}\sum \frac{Q(\chi_1)+Q(\chi_2)}{\varphi(q_1)\varphi(q_2)}.$$

This Q can be estimated as

$$Q \ll 1 \tag{31}$$

by recalling Landau's theorem on the exceptional characters. See Corollary 11.8 of [10]. Substituting (31) into (30), we arrive at Theorem 1.

Acknowledgments

First, the author would like to express his gratitude to Prof. Yuichi Kamiya and Prof. Hideaki Ishikawa for giving the opportunity to give a talk. Second, the author also would like to express his gratitude to Prof. Kohji Matsumoto for his suggestion of this problem, advice and encouragement. Finally, the author would like to express his gratitude to Prof. Alessandro Languasco for pointing out some inaccuracies in my original preprint.

References

[1] G. Bhowmik and J-C. Schlage-Puchta, Mean representation number of integers as the sum of primes, *Nagoya Math. Journal* **200** (2010) 27–33.

[2] H. Cramér, Some theorems concerning prime numbers, *Arkiv för Mat. Astr. Fys.* **15** (5), 33 pp; *Collected works*, Vol. 1, (Springer-Verlag, 1994), pp. 138–170.

[3] S. Egami and K. Matsumoto, Convolutions of the von Mangoldt function and related Dirichlet series, *Proceedings of the 4th China-Japan Seminar held at Shangdong*, ed. S. Kanemitsu and J.-Y. Liu, (World Sci. Publ., 2007), pp. 1–23.

[4] A. Fujii, An additive problem of prime numbers, *Acta Arith.* **58** (1991) 173–179.

[5] A. Fujii, An additive problem of prime numbers. II, *Proc. Japan Acad. Ser. A Math. Sci.* **67** (1991) 248–252.

[6] P. X. Gallagher, A large sieve density estimate near $\sigma = 1$, *Invent. Math.* **11** (1970) 329–339.

[7] D. A. Goldston and Liyang Yang, The average number of Goldbach representations, preprint (2016), `arXiv:1601.06902`.

[8] A. Languasco and A. Perelli, On Linnik's theorem on Goldbach numbers in short intervals and related problems, *Ann. Inst. Fourier* **44** **(2)** (1994) 307–322.

[9] A. Languasco and A. Zaccagnini, The number of Goldbach representations of an integer, *Proc. Amer. Math. Soc.* **140** (2012) 795–804.

[10] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, (Cambridge University Press, 2007).

[11] K. Prachar, Generalization of a theorem of A. Selberg on primes in short intervals, *Topics in Number Theory*, Colloq. Math. Soc. János Bolyai **13** (1974), pp. 267–280.

[12] F. Rüppel, *Convolution of the von Mangoldt function over residue classes*, Diplomarbeit, Univ. Würzburg, (2009).

[13] F. Rüppel, Convolution of the von Mangoldt function over residue classes, *Šiauliai Math. Semin.* **7** **(15)** (2012) 135–156.

[14] B. Saffari and R. C. Vaughan, On the fractional parts of x/n and related sequences II, *Ann. Inst. Fourier* **27** **(2)** (1997) 1–30.

[15] A. Selberg, On the normal density of primes in small intervals, and the difference between consecutive primes, *Arch. Math. Naturvid.* **47** (1943) 87–105; Collected Papers, Vol. 1, (Springer Verlag, 1989), pp. 160–178.

[16] Wang Yuan and Shan Zun, A conditional result on Goldbach problem, Acta Math. Sinica, New Series **1** (1) (1985) 72–78.

[17] Y. Suzuki, A mean value of the representation function for the sum of two primes in arithmetic progressions, preprint (2015), arXiv:1504.01967.

Yuta Suzuki
Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya 464-8602, Japan
e-mail: m14021y@math.nagoya-u.ac.jp